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ON THE THERMODYNAMIC FOUNDATIONS OF STRAIN-DEPENDENT CREEP DAMAGE AND RUPTURE IN THREE DIMENSIONS

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Abstract—The laws of thermodynamics are employed to clarify the concepts of nonisothermal multi-dimensional creep damage and rupture. By postulating a Gibbs free energy functional in terms of time integrals of functions of stress, temperature and damage, we deduce via the Second Law of Thermodynamics constitutive laws for strain, entropy and damage which are general enough to encompass various existing theories. Then by choosing a more specific form for the Gibbs free energy we obtain the governing constitutive laws of the strain-dependent theory of damage. The formulation of the boundary value problem for stress, displacement, temperature and damage is also discussed, for both the coupled and quasi-static uncoupled cases.

NOTATION

A_0 initial cross-sectional area
A_r effective undamaged area, eqn (1)
B material damage constant, eqn (5)
C, m multi-dimensional creep constants, eqn (12)
D damage, eqn (4)
D_0 initial damage
D_c critical value of damage at rupture
$\tilde{D}$ damage increment, eqn (8)
e_{ijk} alternating tensor
E Young's modulus
f_k scalar functions of σ_{mn}, $\tilde{T}$ and $\tilde{D}$, eqn (23)
F_i body force per unit volume
g_k tensor functions of σ_{mn}, $\tilde{T}$ and $\tilde{D}$, eqn (23)
G Gibbs free energy functional
I_1 first invariant of stress
J_2 second invariant of stress deviator
k_{ij} conductivity tensor, eqn (28)
K, n creep constants, eqn (6)
N normal coordinate to Σ, eqn (40)
q_i heat efflux vector
Q heat input, eqn (18)
r distributed heat source
s entropy
s_{ij} stress deviator tensor
S surface area
t time
t_{ri} time to initial rupture
T temperature
T_0 reference temperature
$\tilde{T}$ temperature increment, eqn (10)
$T_i^{(n)}$ traction vector
u specific internal potential energy
u_i displacement field
U internal kinetic and potential energy, eqn (19)
$U(\ )$ unit step function
v_i velocity field
V volume
W work input, eqn (17)
x_i rectangular Cartesian coordinates
α, γ, δ material damage powers, eqn (5)

α_T coefficient of thermal expansion
β, ν material damage constants of Kachanov, eqn (3)
δ_{ij} Kronecker delta
ϵ_{ij} strain tensor
ϵ_{cij} creep strain tensor
$\dot{\epsilon}_{c1}$ creep strain rate corresponding to maximum principal stress σ_1, eqn (15)
Λ rate of energy dissipation, eqn (29)
μ damage dissipation energy material constant, eqn (35)
ν_e elastic Poisson's ratio
ρ density
ρ_0 initial density
σ_{ij} stress tensor
$\sigma_1, \sigma_2, \sigma_3$ principal stresses, ordered from maximum to minimum
Σ moving boundary, Fig. 1
τ domain of volume
ψ continuity, eqn (1)
ω Kachanov damage, eqn (2)
$\dot{()}$ indicates $\partial/\partial t$, assuming infinitesimal theory
$\{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k$ indicates products of powers of time integrals of functions of σ_{mn}, $\tilde{T}$ and $\tilde{D}$

1. INTRODUCTION

The importance of creep damage and rupture in structural components at elevated temperatures cannot be overemphasized, since clearly one of the most important requirements of a safe design is that a component not rupture during its useful lifetime. Creep damage and rupture are difficult concepts to define, since complex continuum mechanical and physical metallurgical principles are involved. On the one hand there are approaches based primarily on macroscopic continuum mechanical postulates; among these is the ductile rupture theory due to Hoff[1], the distributed damage and brittle rupture theory due to Kachanov[2], and various subsequent improvements such as the theory due to Rabotnov[3] which attributes tertiary creep to damage. On the other hand there are approaches based primarily on microscopic physical metallurgical observations associated with damage, such as variations of sound speed and electrical resistivity, neutron and X-ray diffraction studies, and actual void counting with the aid of optical and electron microscopes. Recently, Piatti *et al.*[4] have developed very sensitive differential density measurement techniques as an index of damage due to void nucleation and growth. Using data obtained for steel with such density measurement techniques, Belloni *et al.*[5, 6] combined the continuum mechanical and physical metallurgical approaches to develop a strain-dependent theory of damage which is essentially a modification of Kachanov's theory. This initial strain-dependent model has been extended to more general loads and materials by Belloni, Bernasconi, Cozzarelli, Lee and Piatti[7–9].

It is our major goal here to employ the laws of thermodynamics to help clarify the concepts of non-isothermal creep damage and rupture in three dimensions, and to gain useful conditions on the constitutive laws of damage. We use the energy functional approach (see [10, 11]) in a manner similar to that employed by Chang and Cozzarelli[12] for nonlinear thermoviscoelastic materials, and by Cozzarelli and Huang[13] for materials undergoing thermally and irradiation induced creep. We will not employ the internal variable approach (see [14]) which was used by Chaboche[15] and Lemaitre and Chaboche[16] to study damage, but it is well recognized that the two approaches are fully consistent. We also address the question of proper formulation of the boundary value problem for stress, displacement, temperature and damage, and make some brief observations concerning uniqueness using the results given in [17]. Although most of our attention is directed toward the strain-dependent theory of damage developed in [5–9], our results are also applicable to other theories of damage (e.g. [2, 3]).

In Section 2 we discuss the strain-dependent theory of creep damage for inhomogeneous materials subjected to time-dependent tensile or compressive stress histories, with special attention paid to the formulation for multiaxial stress states with certain material parameters dependent on temperature which in turn may vary in space and time. Then in Section 3 we postulate a Gibbs free energy functional containing products of powers of time integrals of functions of stress, temperature increment and damage increment, and we are able to deduce via the first and second laws of thermodynamics three dimensional constitutive laws for the strain and entropy in explicit form and for the damage increment in implicit form. By choosing more specific forms for the Gibbs free energy and by introducing the concept of a critical damage at which rupture occurs, one may obtain various damage theories from these constitutive laws, and we illustrate this in detail for the strain-dependent creep damage theory discussed in Section 2.

Section 4 begins with a presentation of the general coupled boundary value problem, which consists of a set of 18 simultaneous equations in stress, strain, displacement, temperature, entropy and damage. As in Kachanov[2], we first consider a period of "latent failure" which extends up to the time at which critical damage is reached and rupture begins, and then a period of "propagation of failure" which continues up to time at which the material is no longer able to support the applied loads. During the first time period we have a boundary value problem with a fixed boundary, while during the second time period we have a moving boundary and a boundary value problem with features similar to the Stefan problem in heat conduction[18]. Section 4 then concludes with a presentation of the uncoupled quasi-static stress boundary value problem, including the governing stress compatibility field equations and some brief comments on uniqueness of solution.

2. STRAIN-DEPENDENT THEORY OF CREEP DAMAGE

In the study of creep rupture the concept of damage has been introduced in various ways. The most widely used approach is due to Kachanov[2], who used simple ideas from the one-dimensional tensile test to define a quantity called the continuity as

$$\psi(t) = \frac{A_r(t)}{A_0} \tag{1}$$

where A_0 is the initial cross sectional area of the test specimen, and $A_r(t)$ is the effective undamaged area capable of resisting the load at any instant. The "damage" $\omega(t)$ was then introduced in terms of the continuity as

$$\omega(t) = 1 - \psi(t) = \frac{A_0 - A_r(t)}{A_0} \tag{2}$$

which was defined as running from 0 to 1, as the material underwent transition from the initial undamaged state ($A_r = A_0$) to the final state of critical damage and rupture ($A_r = 0$). Kachanov also postulated that the rate of damage increase in accordance with the one-dimensional power law

$$\dot{\omega}(t) = \beta\left[\frac{\sigma(t)}{1-\omega(t)}\right]^{\nu} \tag{3}$$

where σ is the tensile stress and β and ν are material damage constants. Note that the damage as given by eqn (3) is not directly strain-dependent.

Belloni *et al.*[5, 6] attempted to estimate the quantities appearing in eqn (3) by measuring density variation[4] and defining the damage as the dimensionless quantity

$$D(t) = -\frac{\Delta\rho}{\rho_0} = \frac{\rho_0 - \rho(t)}{\rho_0} \tag{4}$$

where ρ_0 is the initial density of the undamaged material and $\rho(t)$ is the density at time t. In analogy with the Kachanov damage ω, the damage $D(t)$ was defined as running from 0 to a critical value at rupture, D_r, which is a material constant. These experiments indicated that the creep strain ϵ_c has a very significant effect on the accumulation of damage, and good agreement was obtained with data for the case of one dimensional constant tensile stress σ_0 using the strain-dependent power law

$$D(t) = B\epsilon_c(t)^{\alpha}\sigma_0^{\gamma}t^{\delta} \tag{5}$$

where B, α, γ, δ are positive material damage constants at a particular constant temperature. If we neglect transient creep, the creep strain is given by the Norton power law

$$\epsilon_c(t) = K\sigma_0{}^n t \tag{6}$$

where K, n are positive material constants at constant temperature.

Cozzarelli and Bernasconi[7] extended eqn (5) to the case of one-dimensional variable tensile stress. Lee and Cozzarelli[9] extended it further to compressive as well as tensile one dimensional stress, and also to the case of inhomogeneous materials. Equations (5) and (6) then extend to

$$\tilde{D}(x,t) = B(x)\left\{K(x)\int_0^t \sigma(x,t')^n U[\sigma(x,t')]\,dt'\right\}^\alpha \left\{\int_0^t \sigma(x,t')^{\gamma/\delta} U[\sigma(x,t')]\,dt'\right\}^\delta \tag{7a}$$

$$\epsilon_c(x,t) = K(x)\int_0^t \sigma(x,t')^n\,dt' \tag{7b}$$

where $\sigma(x,t)$ may now vary with position and time. In eqn (7a) the first term in brackets is the creep strain, $U(\sigma)$ is the unit step function which was introduced to reproduce the experimentally observed result that in many materials little damage is produced under compression, $B(x)$ and $K(x)$ (but not the powers α, γ, δ, n) may vary with position due to material inhomogeneity, and

$$\tilde{D}(x,t) = D(x,t) - D_0(x) \tag{8}$$

where $D_0(x)$ is an initial inhomogeneous state of damage. The lower limits in eqns (7) have been set at 0, since $\sigma(t)$ is assumed to vanish for $t < 0$. As previously noted, for simplicity transient creep was not included in eqns (5)–(7), and thus the stress in eqns (7) is constrained to vary *slowly* with time. For more rapid loading conditions such as in the case of cyclic variation of stress, it would not only be necessary to employ a more general creep law (e.g. see [12, 19]) but it might also be necessary to include fatigue damage in addition to creep damage (see [15]).

It is interesting to note that if eqn (7a) is restricted to homogeneous materials subjected to tensile creep only, then this strain dependent damage law can be expressed in a form similar to the integral of Kachanov's law (3) for the special case where $\gamma = n\delta$ in (7a) and $v = n$ in (3). Accordingly, we obtain

$$1 - [1 - \omega(t)]^{n+1} = \beta(n+1)\int_0^t \sigma^n(t')\,dt' \tag{9a}$$

$$D(t)^{1/(\alpha+\delta)} = (BK^\alpha)^{1/(\alpha+\delta)}\int_0^t \sigma^n(t')\,dt' \tag{9b}$$

which establishes an analogy between the two formulations under these special conditions, if we require in addition that $BK^\alpha = [\beta(n+1)]^{\alpha+\delta}$ and $D = [1 - (1-\omega)^{n+1}]^{\alpha+\delta}$.

In this paper we take one-dimensional eqns (7) one step further and permit parameters B and K to also vary with the space and time dependent temperature increment

$$\tilde{T}(x,t) = T(x,t) - T_0(x) \tag{10}$$

where $T_0(x)$ is a reference temperature. By following the same procedure outlined in [7, 9] in obtaining eqns (7), we now get

$$\tilde{D}(x,t) = \left\{\int_0^t K[x,\tilde{T}(x,t')]\sigma(x,t')^n U[\sigma(x,t')]\,dt'\right\}^\alpha$$

$$\times \left\{ \int_0^t B[x, \tilde{T}(x, t')]^{1/\delta} \sigma(x, t')^{\gamma/\delta} U[\sigma(x, t')] \, \mathrm{d}t' \right\}^{\delta} \tag{11a}$$

$$\epsilon_c(x, t) = \int_0^t K[x, \tilde{T}(x, t')] \sigma(x, t')^n \, \mathrm{d}t'. \tag{11b}$$

The present study is not limited to the one-dimensional case, and thus it is necessary to present multiaxial extensions of eqns (11). For the creep law we shall employ the isotropic incompressible steady creep law (see [20] for a discussion of compressibility and transient creep) of the Mises-type

$$\epsilon_{cij}(x_i, t) = \int_0^t C[x_i, \tilde{T}(x_i, t')] J_2{}^m s_{ij} \, \mathrm{d}t' \tag{12}$$

where $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$ is the stress deviator and $J_2 = \frac{1}{2}s_{ij}s_{ij}$ is the second invariant of the stress deviator, and where

$$m = \frac{n-1}{2} \qquad C = \frac{3^{(n+1)/2}K}{2}. \tag{13}$$

In selecting a multiaxial damage law we adopt the simplest approach, and assume that damage in three dimensions is a scalar computed from the maximum principal stress (see [21]) and, for strain-dependent damage, from the corresponding creep strain. Accordingly, eqn (11a) extends to

$$\tilde{D}(x_i, t) = \left\{ \int_0^t \dot{\epsilon}_{c1}(x_i, t') U(\sigma_1) \, \mathrm{d}t' \right\}^{\alpha} \left\{ \int_0^t B[x_i, \tilde{T}(x_i, t')]^{1/\delta} \sigma_1(x_i, t')^{\gamma/\delta} U(\sigma_1) \, \mathrm{d}t' \right\}^{\delta} \tag{14}$$

where σ_1 is the maximum of the principal stresses σ_1, σ_2, σ_3 and $\dot{\epsilon}_{c1}$ is the corresponding creep strain rate obtained from eqn (12) as

$$\dot{\epsilon}_{c1} = \frac{C[x_i, \tilde{T}(x_i, t)]}{2^m 3^{m+1}} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^m (2\sigma_1 - \sigma_2 - \sigma_3). \tag{15}$$

For a discussion of more general multi-axial damage laws, see [9, 15] and the work of Martin and Leckie[22], and Hayhurst and Leckie[23]. In the next section we shall prove that eqn (14) is consistent with the laws of thermodynamics.

3. THERMODYNAMIC BASIS OF DAMAGE

(a) *First and second laws of thermodynamics*

Consider a continuum subjected to work input W and heat input Q on its surface and throughout its volume. This energy input is converted into change of the internal kinetic and potential energy U, associated not only with the velocity v_i, strain ϵ_{ij} and temperature T fields, but also with the damage field D due to the creation of new internal surface area at voids. The usual global form of the first law of thermodynamics is thus expressed as

$$\dot{W} + \dot{Q} = \dot{U} \tag{16}$$

where a dot indicates time rate of change.

For infinitesimal theory, $\dot{W}$, $\dot{Q}$ and $\dot{U}$ are written in the usual manner (see [24]) as volume integrals over the domain τ. Thus

$$\dot{W} = \int_\tau F_i v_i \, \mathrm{d}V + \int_\tau \frac{\partial v_i}{\partial x_j} \sigma_{ij} \, \mathrm{d}V + \int_\tau v_i \frac{\partial \sigma_{ij}}{\partial x_j} \, \mathrm{d}V \tag{17}$$

$$\dot{Q} = -\int_\tau \frac{\partial q_i}{\partial x_i} \, \mathrm{d}V + \int_\tau \rho_0 r \, \mathrm{d}V \tag{18}$$

$$\dot{U} = \frac{\partial}{\partial t}\left(\int_{\tau} \frac{1}{2}\rho_0 v_i v_i \, dV + \int_{\tau} \rho_0 u \, dV\right) \tag{19}$$

where F_i is the body force per unit volume, $\partial q_i/\partial x_i$ is the divergence of the rate of heat efflux vector q_i, ρ_0 is the mass density of the undamaged material and thus assumed constant, r is the strength of a distributed rate of heat source, and u is the specific internal potential energy.

Combining eqns (16)–(19) and introducing the Gibbs free energy $G = (\sigma_{ij}\epsilon_{ij}/\rho_0) - u + Ts$ where s is the entropy, the local form of the first law of thermodynamics is obtained as

$$-\epsilon_{ij}\frac{\partial \sigma_{ij}}{\partial t} - \frac{\partial q_i}{\partial x_i} + \rho_0 r = \rho_0\left(-\frac{\partial G}{\partial t} + s\frac{\partial T}{\partial t} + T\frac{\partial s}{\partial t}\right) \tag{20}$$

with the use of the equations of motion. The second law of thermodynamics is expressed as the usual inequality[24]

$$\rho_0 T\frac{\partial s}{\partial t} - \rho_0 r + \frac{\partial q_i}{\partial x_i} - \frac{q_i}{T}\frac{\partial T}{\partial x_i} \geq 0. \tag{21}$$

Finally, combining (20) and (21) and introducing the definition of the temperature increment relative to an initial inhomogeneous state (see eqn 10), we obtain for infinitesimal theory

$$-\epsilon_{ij}\frac{\partial \sigma_{ij}}{\partial t} + \rho_0\frac{\partial G}{\partial t} - \rho_0 s\frac{\partial \tilde{T}}{\partial t} - \frac{q_i}{T_0}\frac{\partial \tilde{T}}{\partial x_i} \geq 0. \tag{22}$$

In this inequality we take σ_{ij}, $\tilde{T}$ and the damage increment $\tilde{D}$ (see eqn 8) as the basic thermodynamic variables, and for convenience $\tilde{D}$ is taken to be dimensionless.

(b) *Constitutive equations of damage*

We postulate for the Gibbs free energy functional the expression

$$G = \sum_k [f_k(\sigma_{mn}, \tilde{T}, \tilde{D}) + g_k(\sigma_{mn}, \tilde{T}, \tilde{D})\{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k] \tag{23}$$

where f_k and g_k are scalar and tensor functions respectively of σ_{mn}, $\tilde{T}$ and $\tilde{D}$, and where the notation $\{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k$ is used to represent products of powers of time integrals of functions of σ_{mn}, $\tilde{T}$ and $\tilde{D}$. By postulating G in this general form, one may obtain via thermodynamics not only the law of strain-dependent damage (eqn 14) but also other damage laws such as due to Kachanov[2] and Rabotnov[3].

Differentiating eqn (23) we obtain

$$\frac{\partial G}{\partial t} = \sum_k \left[\left(\frac{\partial f_k}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f_k}{\partial \tilde{T}}\dot{\tilde{T}} + \frac{\partial f_k}{\partial \tilde{D}}\dot{\tilde{D}}\right) + \left(\frac{\partial g_k}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial g_k}{\partial \tilde{T}}\dot{\tilde{T}} + \frac{\partial g_k}{\partial \tilde{D}}\dot{\tilde{D}}\right)\right.$$

$$\left. \times \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k + \overline{\{\sigma_{mn}, \tilde{T}, \tilde{D}\}}^{\,\cdot}_k g_k\right] \tag{24}$$

where $(\dot{\ })$ represents partial differentiation with respect to time. Substituting (24) into inequality (22) and rearranging terms we obtain

$$\left[-\epsilon_{ij} + \rho_0\sum_k\left(\frac{\partial f_k}{\partial \sigma_{ij}} + \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k\frac{\partial g_k}{\partial \sigma_{ij}}\right)\right]\dot{\sigma}_{ij} + \left[-\rho_0 s + \rho_0\sum_k\left(\frac{\partial f_k}{\partial \tilde{T}} + \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k\frac{\partial g_k}{\partial \tilde{T}}\right)\right]\dot{\tilde{T}}$$

$$+ \rho_0\sum_k\left(\frac{\partial f_k}{\partial \tilde{D}} + \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k\frac{\partial g_k}{\partial \tilde{D}}\right)\dot{\tilde{D}} + \rho_0\sum_k\overline{\{\sigma_{mn}, \tilde{T}, \tilde{D}\}}^{\,\cdot}_k g_k - \frac{q_i}{T_0}\frac{\partial \tilde{T}}{\partial x_i} \geq 0. \tag{25}$$

In inequality (25) we first set the coefficients of the basic thermodynamic variables $\dot{\sigma}_{ij}$, $\dot{\tilde{T}}$, $\dot{\tilde{D}}$ equal to zero, and thereby obtain the constitutive relations

$$\epsilon_{ij} = \rho_0 \sum_k \left(\frac{\partial f_k}{\partial \sigma_{ij}} + \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k \frac{\partial g_k}{\partial \sigma_{ij}} \right) \tag{26a}$$

$$s = \sum_k \left(\frac{\partial f_k}{\partial \tilde{T}} + \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k \frac{\partial g_k}{\partial \tilde{T}} \right) \tag{26b}$$

$$0 = \sum_k \left(\frac{\partial f_k}{\partial \tilde{D}} + \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k \frac{\partial g_k}{\partial \tilde{D}} \right). \tag{26c}$$

Note that eqn (26c) will in general yield an implicit relation in $\tilde{D}$, the damage increment variable. It then follows that for inequality (25) to be satisfied it is necessary that

$$\rho_0 \sum_k \overline{\{\sigma_{mn}, \dot{\tilde{T}}, \tilde{D}\}}_k g_k - \frac{q_i}{T_0} \frac{\partial \tilde{T}}{\partial x_i} \geq 0. \tag{27}$$

If we introduce the Fourier heat conduction law

$$q_i = -k_{ij} \frac{\partial \tilde{T}}{\partial x_j} \tag{28}$$

where k_{ij} is the positive definite conductivity tensor, and define the rate of energy dissipation as

$$\Lambda = \rho_0 \sum_k \overline{\{\sigma_{mn}, \dot{\tilde{T}}, \tilde{D}\}}_k g_k \tag{29}$$

inequality (27) becomes

$$\Lambda + \frac{k_{ij}}{T_0} \frac{\partial \tilde{T}}{\partial x_j} \frac{\partial \tilde{T}}{\partial x_i} \geq 0. \tag{30}$$

Since inequality (30) must hold for all values of $(\partial \tilde{T}/\partial x_i)$, including the particular case $(\partial \tilde{T}/\partial x_i) = 0$, it is necessary that

$$\Lambda \geq 0. \tag{31}$$

3(c). *Special case—strain dependent damage law*

For the sake of brevity we shall not display the arguments x_i in this section. In expression (23) for the Gibbs free energy functional we shall choose

$$g_1(\sigma_{mn}, \tilde{T}, \tilde{D}) = \sigma_{mn}, \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_1 = \frac{1}{\rho_0} \int_0^t C(\tilde{T}(t')) J_2(t')^n s_{mn}(t')\, dt' \tag{32a}$$

where all quantities were defined in eqns (12)–(13), and

$$f_1(\sigma_{mn}, \tilde{T}, \tilde{D}) = \frac{1}{\rho_0} \left[\frac{1+\nu_e}{2E} \left(\sigma_{pq}\sigma_{pq} - \frac{\nu_e}{1+\nu_e} I_1^2 \right) + \alpha_T \tilde{T} I_1 \right] \tag{32b}$$

where ν_e, E are the elastic Poisson's ratio and Young's modulus respectively, $I_1 = \sigma_{pp}$ is the first invariant of stress, and α_T is the coefficient of thermal expansion. Then, since we are not concerned with the form of the entropy law in this paper, we shall simply set

$$f_2(\sigma_{mn}, \tilde{T}, \tilde{D}) = g(\tilde{T}), \qquad g_2(\sigma_{mn}, \tilde{T}, \tilde{D}) = 0. \tag{32c}$$

where $g(\tilde{T})$ is some unspecified function of temperature increment. Lastly, we select

$$f_3(\sigma_{mn}, \tilde{T}, \tilde{D}) = -\frac{\mu}{2\rho_0}\tilde{D}^2, \quad g_3(\sigma_{mn}, \tilde{T}, \tilde{D}) = \frac{\mu}{\rho_0}\tilde{D}$$

$$\{\sigma_{mn}, \tilde{T}, \tilde{D}\}_3 = \left[\int_0^t \dot{\epsilon}_{c1}(t')U(\sigma_1(t'))\,\mathrm{d}t'\right]^\alpha \left[\int_0^t B(\tilde{T}(t'))^{1/\delta}\sigma_1(t')^{\gamma/\delta}U(\sigma_1(t'))\,\mathrm{d}t'\right]^\delta \tag{32d}$$

where μ is a material constant whose physical meaning we shall soon discuss, and all other quantities were defined in eqns (14) and (15).

It then follows from eqns (26c) and (32) that the damage law is given as

$$\tilde{D}(t) = \left[\int_0^t \dot{\epsilon}_{c1}(t')U(\sigma_1(t'))\,\mathrm{d}t'\right]^\alpha \left[\int_0^t B(\tilde{T}(t'))^{1/\delta}\sigma_1(t')^{\gamma/\delta}U(\sigma_1(t'))\,\mathrm{d}t'\right]^\delta \tag{33}$$

which is the same strain-dependent damage law presented in Section 2 (eqn 14). Furthermore, it follows from eqns (26a) and (32) that the strain is given by

$$\epsilon_{ij} = \frac{1+\nu_e}{E}\left(\sigma_{ij} - \frac{\nu_e}{1+\nu_e}I_1\delta_{ij}\right) + \alpha_T\tilde{T}\delta_{ij} + \int_0^t C(\tilde{T}(t'))J_2(t')^m s_{ij}(t')\,\mathrm{d}t \tag{34}$$

which is the isotropic stress power law for the incompressible steady creep case given in Section 2 (eqn 12), plus the usual isotropic linear thermoelastic terms. In a similar straightforward manner one could obtain various other damage and creep laws.

Finally, the rate of energy dissipation is obtained from eqns (29) and (32) as

$$\Lambda = 2CJ_2^{m+1} + \mu\tilde{D}\dot{\tilde{D}} \tag{35}$$

where the first term is due to the creep strain while the second is due to the damage. Thus we see that μ is a material constant associated with energy dissipation due to damage and with units of energy per unit volume. Since all material constants are assumed to be positive, and J_2 is non-negative by definition, and also $\tilde{D} \geq 0$ and $\dot{\tilde{D}} \geq 0$ because both integrands in eqn (33) are non-negative, it follows that each term in eqn (35) is non-negative. Accordingly, $\Lambda \geq 0$ and thus the present formulation is consistent with the second law of thermodynamics.

In the next section we discuss the formulation of the boundary value problem.

4. FORMULATION OF THE BOUNDARY VALUE PROBLEM

(a) *Complete set of equations in general coupled problem*

With the use of eqns (23), (26) and (28), coupled energy equation (20) may be rewritten in the compact form

$$\rho_0 r + k_{ij}\frac{\partial^2\tilde{T}}{\partial x_i \partial x_j} + \rho_0\sum_k \overline{\{\sigma_{mn}, \tilde{T}, \tilde{D}\}}^{\boldsymbol{\cdot}}_k g_k - \rho_0 T\dot{s} = 0 \tag{36}$$

where we recognize the third term as the energy dissipation Λ (eqn 29). For convenience we reproduce constitutive equations (26) below:

$$\epsilon_{ij} = \rho_0\sum_k\left(\frac{\partial f_k}{\partial\sigma_{ij}} + \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k\frac{\partial g_k}{\partial\sigma_{ij}}\right) \tag{37a}$$

$$s = \sum_k\left(\frac{\partial f_k}{\partial\tilde{T}} + \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k\frac{\partial g_k}{\partial\tilde{T}}\right) \tag{37b}$$

$$0 = \sum_k\left(\frac{\partial f_k}{\partial\tilde{D}} + \{\sigma_{mn}, \tilde{T}, \tilde{D}\}_k\frac{\partial g_k}{\partial\tilde{D}}\right). \tag{37c}$$

Finally, we add the usual equations of motion and strain displacement relations of infinitesimal theory

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = \rho_0 \ddot{u}_i \tag{38a}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{38b}$$

where u_i is the displacement field and F_i is the body force.

Equations (36)–(38) are a set of 18 simultaneous equations in the variables σ_{ij}, ϵ_{ij}, u_i, $\tilde{T}$, $\tilde{D}$ and s, which as described in Kachanov[2] are subjected to two sets of boundary and initial conditions for two different intervals of time. Following [2], we designate the first interval as the period of "latent failure" defined by $0 \le t \le t_{ri}$, where the load is applied at $t = 0$ and t_{ri} is the time at which the total damage $D = \tilde{D} + D_0$ (see eqn 8) first reaches (at some point) the critical value at rupture, D_r. During this period of latent failure one typically prescribes tractions on the boundary, i.e.

$$\sigma_{ij} n_j = T_i^{(n)}(x_i, t) \text{ on } S \tag{39}$$

where x_i are coordinates on the fixed boundary S and $T_i^{(n)}$ is a given traction vector.

At time t_{ri} a failure front begins to propagate through the material, and again following [2] we designate this second time interval $t \ge t_{ri}$ as the period of "propagation of failure". During this time period we have a moving boundary value problem somewhat similar to the Stefan problem in heat conduction. This is shown schematically in Fig. 1, where $\Sigma(t)$ is a moving boundary on which $D = D_r$, and domain $\tau_1(t)$ is assumed to have no load-carrying capacity, i.e. it has "ablated" away. Equations (36)–(38) must now be solved simultaneously for domain $\tau_2(t)$, where tractions are prescribed on $S_2(t)$, and, using the fact that the damage is a prescribed constant on $\Sigma(t)$, one may obtain (see [2]) the boundary condition

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{\partial D}{\partial t} \bigg/ \frac{\partial D}{\partial N} \text{ on } \Sigma(t) \tag{40}$$

where N is the normal coordinate to $\Sigma(t)$.

(b) *Uncoupled quasi-static problem with strain-dependent damage*

For simplicity, let us neglect thermomechanical coupling, body forces and inertia, and furthermore employ the strain-dependent damage law and creep law given in Section 3(c). Accordingly, the temperature is assumed prescribed and eqns (36) and (37b) are no longer

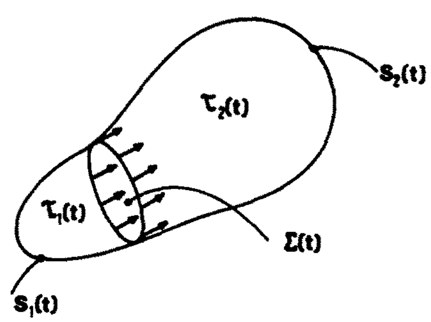

Fig. 1. Period of propagation of failure.

required, and eqns (38a), (38b), (37a) and (37c) become respectively

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{41a}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{41b}$$

$$\epsilon_{ij} = \frac{1+\nu_e}{E}\left(\sigma_{ij} - \frac{\nu_e}{1+\nu_e} I_1 \delta_{ij}\right) + \alpha_T \tilde{T} \delta_{ij} + \int_0^t C(\tilde{T}(t'))J_2(t')^m s_{ij}(t')\,\mathrm{d}t' \tag{41c}$$

$$\tilde{D}(t) = \left[\int_0^t \dot{\epsilon}_{c1}(t')U(\sigma_1(t'))\,\mathrm{d}t'\right]^{\alpha}\left[\int_0^t B(\tilde{T}(t'))^{1/\delta}\sigma_1(t')^{\gamma/\delta}U(\sigma_1(t'))\,\mathrm{d}t'\right]^{\delta}. \tag{41d}$$

Equations (41a–c) are 15 equations in σ_{ij}, ϵ_{ij} and u_i which may be solved first with the given boundary conditions, and then the results may be substituted into eqn (41d) to compute the damage in order to keep track of the propagation of failure as discussed in the previous section. Thus, in the uncoupled problem not only the temperature but also the damage is uncoupled from the simultaneous field equations.

Assuming tractions are prescribed, we may, of course, obtain a stress formulation from eqns (41a–c) with the use of the compatibility equations

$$e_{ikm} e_{jln} \frac{\partial^2 \epsilon_{kl}}{\partial x_m \partial x_n} = 0 \tag{42}$$

where e_{ijk} is the alternating tensor. Proceeding in the usual manner (e.g. see [25]), we obtain the stress compatibility equations

$$(1+\nu_e)\frac{\partial^2 \sigma_{ij}}{\partial x_k \partial x_k} + \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} + E\alpha_T\left(\frac{1+\nu_e}{1-\nu_e}\frac{\partial^2 \tilde{T}}{\partial x_k \partial x_k}\delta_{ij} + \frac{\partial^2 \tilde{T}}{\partial x_i \partial x_j}\right)$$
$$+ E\left(\frac{\partial^2 \epsilon_{cij}}{\partial x_k \partial x_k} - \frac{\partial^2 \epsilon_{cjk}}{\partial x_i \partial x_k} - \frac{\partial^2 \epsilon_{cik}}{\partial x_j \partial x_k} - \frac{\nu_e}{1-\nu_e}\frac{\partial^2 \epsilon_{ckl}}{\partial x_k \partial x_l}\delta_{ij}\right) = 0 \tag{43}$$

where ϵ_{cij} is the creep strain

$$\epsilon_{cij} = \int_0^t C(\tilde{T}(t'))J_2(t')^m s_{ij}(t')\,\mathrm{d}t' \tag{44}$$

and where we have used the identity $\epsilon_{ckk} = 0$. The solution procedure requires solving compatibility equations (43), supplemented by equilibrium equations (41a) and the given boundary tractions, for the stress σ_{ij}; the creep strain and the damage then follow from eqns (44) and (41d) respectively.

As discussed in the previous section, there will be a period of latent failure, with a fixed boundary, followed by a period of propagation of failure, with a moving boundary. For the period of latent failure we have the same type of problem for which uniqueness of solution was proven in [17]. The uniqueness problem for the period of propagation of failure is much more difficult, and we shall not attempt to consider this problem here.

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